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On shell membranes of Enneper type: generalized Dupin cyclides

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Abstract
It is demonstrated that a class of generalized Dupin cyclides arises naturally out of a classical system of equilibrium equations for shell membranes. This class consists of all families of parallel canal surfaces on which the lines of curvature are planar. Various examples of viable membrane geometries such as particular L-minimal surfaces are presented.

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1. Introduction

The geometric analysis of shell membranes which are in equilibrium constitutes an intrinsic part of the theory of thin shells and has a long history (see, e.g., [1] and references therein). However, only recently has it been shown [2] that the classical equilibrium equations associated with a large class of shell membranes are, in fact, integrable and therefore amenable to the tools of modern soliton theory [3, 4]. The latter connection has been used in [5] to investigate shell membranes which admit non-unique stress distributions. As a result, a connection with the classical Lamé equation [6] has been revealed.

The classification of surfaces which possess one or two families of planar lines of curvature has likewise been a subject of interest for more than a century (see, e.g., [7–10]). It is therefore natural to examine shell membranes which assume the shape of Enneper surfaces [11], that is surfaces on which there exists one family of planar lines of curvature. Here, we are concerned with the particular case of shell membranes on which all lines of curvature are planar. It turns out that this geometric problem is closely related to the above-mentioned physical problem of the existence of non-unique stress distributions and may indeed be addressed by means of the techniques developed in [5].

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The analysis of shell membranes of Enneper type is shown to lead to a class of surfaces which one may term 'generalized Dupin cyclides'. Dupin cyclides are surfaces on which all lines of curvature constitute circles [10]. Classical Dupin cyclides not only arise in differential geometry but also in both computer-aided design and in soliton theory. Thus, substantial accounts of the application of Dupin cyclides in solid modelling may be found in [12, 13]. Of recent work one may cite *inter alia* [14] which contains an extensive literature on the subject and [15, 16] on supercyclides.

On the other hand, Dupin cyclides arise in soliton theory as particular 'soliton surfaces' (see, e.g., [17]). Indeed, they are embedded in a more general class of integrable surfaces, namely isothermic surfaces (see [4] and references therein). Thus, classical solitonic equations such as the sine-Gordon, Korteweg–de Vries and nonlinear Schrödinger equations arise in a natural manner out of the Gauss–Mainardi–Codazzi equations for special soliton surfaces which admit invariance under Bäcklund transformations [4, 18]. This invariance generically induces a nonlinear superposition principle for the soliton equation whereby, on iterative application, multi-solitons may be generated. In the classical literature on the geometry of surfaces, such nonlinear superposition principles are known as permutability theorems and originate in the work of Bianchi [19] on pseudospherical surfaces. If a Bäcklund transformation is known for a class of surfaces Σ then a Bäcklund transformation is naturally induced for the class of parallel surfaces.

Interestingly, parallel Dupin cyclide configurations arise naturally in liquid crystal and shell membrane equilibrium states. Thus, in now classical experiments by Friedel [20], it was observed that smectic A liquid crystals can adopt geometric configurations comprised of parallel layers of Dupin cyclides. Indeed, even the confocal conics associated with such Dupin cyclides are manifest in such experiments. The geometric aspects of Friedel's pioneering work were elaborated upon by Bragg [21]. Subsequently, Kléman [22] proposed a theoretical liquid crystal model based on the classical Love equations of elastic membrane theory [23]. Remarkably, this nonlinear liquid crystal model system has been shown to admit parallel cyclide geometries consistent with Friedel's original empirical observations [24]. A particular reduction of the Kléman model delivers a classical shell membrane system as set down in [1] and it is this system with which we are concerned.

2. The shell membrane equilibrium equations: offset membranes

Here, we consider the classical equilibrium equations of a shell membrane $\Sigma : \boldsymbol{r} = \boldsymbol{r}(\alpha, \beta)$ parametrized in terms of curvature coordinates α, β. Hence, its first and second fundamental forms $\mathrm{I} = \mathrm{d}\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{r}$, $\mathrm{II} = -\mathrm{d}\boldsymbol{N} \cdot \mathrm{d}\boldsymbol{r}$ are given by

$$\mathrm{I} = A_1^2\,\mathrm{d}\alpha^2 + A_2^2\,\mathrm{d}\beta^2, \qquad \mathrm{II} = \kappa_1 A_1^2\,\mathrm{d}\alpha^2 + \kappa_2 A_2^2\,\mathrm{d}\beta^2, \tag{1}$$

where $\boldsymbol{N}$ is the unit normal and κ_1, κ_2 denote the principal curvatures. The Gauss–Mainardi–Codazzi equations adopt the form

$$\begin{aligned} &\kappa_{2\alpha} + (\ln A_2)_\alpha(\kappa_2 - \kappa_1) = 0, \\ &\kappa_{1\beta} + (\ln A_1)_\beta(\kappa_1 - \kappa_2) = 0, \\ &\left(\frac{A_{2\alpha}}{A_1}\right)_\alpha + \left(\frac{A_{1\beta}}{A_2}\right)_\beta + \kappa_1\kappa_2 A_1 A_2 = 0, \end{aligned} \tag{2}$$

while the classical shell membrane equilibrium equations, in the absence of in-plane shear (with respect to lines of curvature), reduce to [1]

$$\begin{aligned}
&T_{1\alpha} + (\ln A_2)_\alpha (T_1 - T_2) = 0, \\
&T_{2\beta} + (\ln A_1)_\beta (T_2 - T_1) = 0, \\
&\kappa_1 T_1 + \kappa_2 T_2 + Z = 0,
\end{aligned} \tag{3}$$

where T_1 and T_2 are the relevant stress resultants and Z is the non-vanishing constant normal loading. It is observed that the assumption of vanishing in-plane shear is equivalent to the requirement that the lines of principal stress on the shell membranes coincide with the lines of curvature. The latter assumption is of significance in, for instance, liquid crystal theory and the theory of biological membranes (see [24, 25] and references therein). In the following, it is understood that the term 'membrane' incorporates the above-mentioned assumptions.

In terms of the variables

$$\begin{aligned}
&p = \frac{A_{1\beta}}{A_2}, \qquad H = A_1, \qquad H_0 = -\kappa_1 A_1, \qquad \tilde{H} = A_1 T_2, \\
&q = \frac{A_{2\alpha}}{A_1}, \qquad K = A_2, \qquad K_0 = -\kappa_2 A_2, \qquad \tilde{K} = A_2 T_1,
\end{aligned} \tag{4}$$

the governing equations (2) and (3) adopt the compact form

$$\mathsf{H}_\beta = p\mathsf{K}, \qquad \mathsf{K}_\alpha = q\mathsf{H}, \qquad p_\beta + q_\alpha + H_0 K_0 = 0 \tag{5}$$

together with the algebraic equation

$$\mathsf{H}^{\mathsf{T}} \Lambda \mathsf{K} = 0, \tag{6}$$

where

$$\mathsf{H} = \begin{pmatrix} H_0 \\ H \\ \tilde{H} \end{pmatrix}, \qquad \mathsf{K} = \begin{pmatrix} K_0 \\ K \\ \tilde{K} \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -Z & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{7}$$

It is readily verified that this system is invariant under the linear transformation

$$(\mathsf{H}\ \mathsf{K}) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ \mathfrak{b} & 1 & 0 \\ \frac{1}{2}\mathfrak{b}^2 Z & \mathfrak{b} Z & 1 \end{pmatrix} (\mathsf{H}\ \mathsf{K}), \tag{8}$$

where $\mathfrak{b} \in \mathbb{R}$ is a real parameter. The Gauss–Mainardi–Codazzi equations (2) guarantee that the Gauss–Weingarten equations

$$\begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \\ \boldsymbol{N} \end{pmatrix}_\alpha = \begin{pmatrix} 0 & -p & -H_0 \\ p & 0 & 0 \\ H_0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \\ \boldsymbol{N} \end{pmatrix}, \qquad \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \\ \boldsymbol{N} \end{pmatrix}_\beta = \begin{pmatrix} 0 & q & 0 \\ -q & 0 & -K_0 \\ 0 & K_0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{X} \\ \boldsymbol{Y} \\ \boldsymbol{N} \end{pmatrix} \tag{9}$$

for the unit tangent vectors $\boldsymbol{X}, \boldsymbol{Y}$ and the normal $\boldsymbol{N}$ to the membrane are compatible. The position vector $\boldsymbol{r}$ to the membrane is then obtained via the pair

$$\boldsymbol{r}_\alpha = H\boldsymbol{X}, \qquad \boldsymbol{r}_\beta = K\boldsymbol{Y}. \tag{10}$$

At the surface level, the transformation (8) induces the invariance

$$\boldsymbol{r} \mapsto \boldsymbol{r} + \mathfrak{b}\boldsymbol{N} \tag{11}$$

corresponding to the transition from a membrane Σ to a parallel (offset) membrane $\Sigma^{\parallel}$ at distance $\mathfrak{b}$.

3. Geometric constraints

The equilibrium equations (3) provide three linear conditions on the two stress resultants T_1, T_2. Thus, the system is over-determined and is compatible only for privileged (families of parallel) shell geometries. In this connection, elimination via differentiation of Z from the membrane system (2) and (3) yields

$$\begin{aligned} T_{1\beta} &= -\left[\ln\left(A_1\kappa_1^2\right)\right]_\beta T_1 + \frac{\kappa_2}{\kappa_1}\left[\ln\left(\frac{A_1}{\kappa_2}\right)\right]_\beta T_2, \\ T_{2\alpha} &= \frac{\kappa_1}{\kappa_2}\left[\ln\left(\frac{A_2}{\kappa_1}\right)\right]_\alpha T_1 - \left[\ln\left(A_2\kappa_2^2\right)\right]_\alpha T_2. \end{aligned} \tag{12}$$

The compatibility conditions $T_{i\alpha\beta} = T_{i\beta\alpha}, i = 1, 2$ on the equilibrium equations (3) are now readily seen to reduce to the single constraint

$$\bar{\mu}T_1 + \bar{\nu}T_2 = 0, \tag{13}$$

where

$$\bar{\mu} = \kappa_1\left(\left[\ln\left(\frac{A_1\kappa_1}{A_2\kappa_2}\right)\right]_{\alpha\beta} + \Upsilon\right), \qquad \bar{\nu} = -\kappa_2\left(\left[\ln\left(\frac{A_1\kappa_1}{A_2\kappa_2}\right)\right]_{\alpha\beta} - \Upsilon\right) \tag{14}$$

and

$$\Upsilon = [\ln(\kappa_1\kappa_2)]_{\alpha\beta} + (\ln A_1)_\beta(\ln\kappa_1)_\alpha + (\ln A_2)_\alpha(\ln\kappa_2)_\beta - (\ln\kappa_1)_\alpha(\ln\kappa_2)_\beta. \tag{15}$$

3.1. L-isothermic membranes

It is evident that, for any admissible membrane geometry, the stress components are uniquely determined by the linear system $(3)_3$, (13) unless $\bar{\mu} = \bar{\nu} = 0$ in which case there exists a one-parameter family of stress distributions. The latter case is represented by the single requirement that

$$\left[\ln\left(\frac{A_1\kappa_1}{A_2\kappa_2}\right)\right]_{\alpha\beta} = 0 \tag{16}$$

since the constraint (13) then reduces to $(\kappa_1T_1 + \kappa_2T_2)\Upsilon = 0$ and hence

$$\Upsilon = 0 \tag{17}$$

by virtue of $(3)_3$ with $Z \neq 0$. Remarkably, the geometric condition (16) is well known in Laguerre geometry and constitutes a defining property of L-isothermic surfaces [26–28]. Thus, we obtain the following theorem.

Theorem 1. *L-isothermic membranes admit a one-parameter family of associated stress distributions.*

Geometrically, an L-isothermic surface may be defined by the requirement that its third fundamental form (spherical representation)

$$\mathrm{III} = \mathrm{d}\boldsymbol{N}\cdot\mathrm{d}\boldsymbol{N} = H_0^2\,\mathrm{d}\alpha^2 + K_0^2\,\mathrm{d}\beta^2 \tag{18}$$

of the surface be conformally flat in terms of appropriately scaled curvature coordinates. Indeed, condition (16) shows that the lines of curvature may be parametrized in such a way that

$$H_0 = K_0 = \mathrm{e}^\theta \tag{19}$$

and hence the third fundamental form of the membrane adopts the form

$$\mathrm{III} = \mathrm{e}^{2\theta}(\mathrm{d}\alpha^2 + \mathrm{d}\beta^2). \tag{20}$$

The Gauss–Mainardi–Codazzi equations then reduce to

$$A_{2\alpha} = \theta_\alpha A_1, \qquad A_{1\beta} = \theta_\beta A_2, \qquad \theta_{\alpha\alpha} + \theta_{\beta\beta} + \mathrm{e}^{2\theta} = 0, \tag{21}$$

where $p = \theta_\beta, q = \theta_\alpha$. In terms of the quantities

$$P = A_1 - A_2, \qquad R = A_1 + A_2, \tag{22}$$

the Mainardi–Codazzi equations $(21)_{1,2}$ may be cast in the form

$$(\mathrm{e}^{-\theta}R)_\alpha = \mathrm{e}^{-2\theta}(\mathrm{e}^{\theta}P)_\alpha, \qquad (\mathrm{e}^{-\theta}R)_\beta = -\mathrm{e}^{-2\theta}(\mathrm{e}^{\theta}P)_\beta, \tag{23}$$

whence the compatibility condition for R yields

$$\mathrm{e}^{-\theta}P_{\alpha\beta} = P(\mathrm{e}^{-\theta})_{\alpha\beta}. \tag{24}$$

Finally, it may be shown that condition (17) is equivalent to [5]

$$(P^2)_{\alpha\beta} = 0 \tag{25}$$

so that P^2 is of the separable form

$$P^2 = f(\alpha) + g(\beta). \tag{26}$$

3.2. Shell membranes of Enneper type

The membrane equations in the case $\bar{\mu} = \bar{\nu} = 0$ have been analysed in detail in [5]. In particular, it has been shown that it is consistent to assume that P depends on one variable only. Here, we demonstrate that the corresponding class of membranes admits a simple geometric interpretation. To this end, it is recalled that the delineation of surfaces on which there exists a family of planar lines of curvature constitutes a classical problem of differential geometry. A special subclass of Enneper surfaces [11] is obtained by demanding that all lines of curvature are planar. Since the torsion of a space curve $\boldsymbol{r} = \boldsymbol{r}(s)$ is given by [10]

$$\tau = \frac{|\boldsymbol{r}', \boldsymbol{r}'', \boldsymbol{r}'''|}{|\boldsymbol{r}' \times \boldsymbol{r}''|^2}, \tag{27}$$

it is readily verified that the conditions for the torsion of the lines of curvature on a surface Σ to vanish read

$$pH_{0\alpha} = p_\alpha H_0, \qquad qK_{0\beta} = q_\beta K_0. \tag{28}$$

Here, we have made use of the Gauss–Weingarten equations (9) of a generic surface parametrized in terms of curvature coordinates α, β. The Gauss–Mainardi–Codazzi equations in the form (5), that is

$$H_{0\beta} = pK_0, \qquad K_{0\alpha} = qH_0, \qquad p_\beta + q_\alpha + H_0K_0 = 0, \tag{29}$$

then imply that

$$\left[\ln\left(\frac{H_0}{K_0}\right)\right]_{\alpha\beta} = 0. \tag{30}$$

Accordingly, the surface Σ is L-isothermic with

$$H_0 = K_0 = \mathrm{e}^{\theta} \tag{31}$$

without loss of generality and the Gauss–Mainardi–Codazzi equations subject to the constraints (28) reduce to the pair

$$\theta_{\alpha\alpha} + \theta_{\beta\beta} + \mathrm{e}^{2\theta} = 0, \qquad (\mathrm{e}^{-\theta})_{\alpha\beta} = 0 \tag{32}$$

with $p = \theta_\beta, q = \theta_\alpha$. Cross-differentiation shows that the constraint $(32)_2$ on the Liouville equation $(32)_1$ is compatible. Theorem 1 therefore implies the following result.

Theorem 2. *Membranes on which the lines of curvature (and therefore the lines of principal stress) are planar admit a one-parameter family of associated stress distributions.*

As stated in the above theorem, the assumption of planar lines of curvature on the membranes considered here implies that $\bar{\mu} = \bar{\nu} = 0$. Accordingly, $\Upsilon = 0$ or, equivalently, $(P^2)_{\alpha\beta} = 0$. By virtue of the compatibility condition (24), the constraint $(32)_2$ shows that $P_{\alpha\beta} = 0$ so that

$$P_\alpha P_\beta = 0. \tag{33}$$

Now, differentiation of the expressions $\kappa_1 = -H_0/H, \kappa_2 = -K_0/K$ and comparison with the relations

$$H_\beta = \theta_\beta K, \qquad K_\alpha = \theta_\alpha H \tag{34}$$

(cf (5)) lead to the equivalence

$$\kappa_{1\alpha}\kappa_{2\beta} = 0 \quad \Leftrightarrow \quad (H_\alpha - \theta_\alpha H)(K_\beta - \theta_\beta K) = 0 \quad \Leftrightarrow \quad P_\alpha P_\beta = 0, \tag{35}$$

where it is recalled that $P = H - K$. Since the condition $\kappa_{1\alpha}\kappa_{2\beta} = 0$ is a defining property of canal surfaces [29, 30], we obtain the following characterization of membranes with planar lines of curvature:

Theorem 3. *The class of membranes on which the lines of curvature (and therefore the lines of principal stress) are planar consists of all canal surfaces with planar lines of curvature. The corresponding fundamental forms parametrized by H, K and H_0, K_0 are obtained via integration of the constrained Liouville equation*

$$\theta_{\alpha\alpha} + \theta_{\beta\beta} = -\mathrm{e}^{2\theta}, \qquad (\mathrm{e}^{-\theta})_{\alpha\beta} = 0 \tag{36}$$

and the compatible Mainardi–Codazzi equations

$$(\mathrm{e}^{-\theta}R)_\alpha = \mathrm{e}^{-2\theta}(\mathrm{e}^\theta P)_\alpha, \qquad P_\alpha P_\beta = 0, \qquad (\mathrm{e}^{-\theta}R)_\beta = -\mathrm{e}^{-2\theta}(\mathrm{e}^\theta P)_\beta \tag{37}$$

with $H_0 = K_0 = \mathrm{e}^\theta$ and $P = H - K, R = H + K$.

Canal surfaces are defined as the envelopes of one-parameter families of spheres of, in general, arbitrary radii and constitute particular Enneper surfaces. Indeed, without loss of generality, canal surfaces are encoded in the constraint $\kappa_{2\beta} = 0$ and it may be shown that the corresponding lines of curvature $\alpha = \text{const}$ constitute circles. If, in addition, $\kappa_{1\alpha} = 0$ then both families of lines of curvature consist of circles and the class of classical Dupin cyclides is obtained [10]. The subclass of canal surfaces defined by the property that all lines of curvature be planar evidently subsumes Dupin cyclides and we therefore refer to these surfaces as generalized Dupin cyclides.

4. Generalized Dupin cyclides

In terms of the variable $\mathrm{e}^{-\theta}$, the solutions of the pair (36) represent separable solutions of the elliptic Liouville equation and may be obtained in a straightforward manner. However, it turns out to be convenient to adopt a linearization procedure which has been developed in [5] in connection with the general case $\bar{\mu} = \bar{\nu} = 0$. Thus, the following theorem may be directly verified:

Theorem 4. *The general solution of the constrained Liouville equation (36) is given by*

$$\mathrm{e}^{-\theta} = \tfrac{1}{2}(|\Phi_1|^2 + |\Phi_2|^2), \tag{38}$$

where $\Phi_1(z)$ *and* $\Phi_2(z)$ *are two solutions of the linear equation*

$$\Phi_{zz} + C\Phi = 0, \qquad z = \alpha + \mathrm{i}\beta \tag{39}$$

subject to the Wronskian condition

$$\Phi_1\Phi_2' - \Phi_1'\Phi_2 = 1. \tag{40}$$

Here, C constitutes an arbitrary real constant.

Specialization to the case of generalized Dupin cyclides of the corresponding theorem set down in [5, theorem 3, p 12] then leads to the following statement.

Theorem 5. *Any solution set* $\{\Phi_1, \Phi_2\}$ *as provided by theorem 4 corresponds to a family of parallel membranes with position vector*

$$\boldsymbol{r} = \mathrm{e}^{-\theta}\lambda_\alpha \boldsymbol{X} + \mathrm{e}^{-\theta}\lambda_\beta \boldsymbol{Y} + (\lambda + \mathfrak{b})\boldsymbol{N}, \tag{41}$$

where λ *is given by*

$$\lambda = \frac{2T_0}{|\Phi_1|^2 + |\Phi_2|^2}, \tag{42}$$

and T_0 *is any particular real solution of the inhomogeneous extension*

$$T_{zz} + CT = \frac{P}{4} \tag{43}$$

of the linear equation (39) with $P_\alpha P_\beta = 0$*. Here,* $\mathfrak{b}$ *is the (real) foliation parameter and the orthonormal frame* $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{N})$ *is determined by*

$$\begin{aligned}\boldsymbol{X} + \mathrm{i}\boldsymbol{Y} &= \frac{1}{|\Phi_1|^2 + |\Phi_2|^2}\begin{pmatrix}\Phi_2^2 - \Phi_1^2\\ \mathrm{i}\left(\Phi_1^2 + \Phi_2^2\right)\\ 2\Phi_1\Phi_2\end{pmatrix},\\ \boldsymbol{N} &= -\frac{1}{|\Phi_1|^2 + |\Phi_2|^2}\begin{pmatrix}\Phi_1\bar{\Phi}_2 + \bar{\Phi}_1\Phi_2\\ \mathrm{i}(\bar{\Phi}_1\Phi_2 - \Phi_1\bar{\Phi}_2)\\ |\Phi_1|^2 - |\Phi_2|^2\end{pmatrix}.\end{aligned} \tag{44}$$

4.1. The position vector

Here, we determine the position vector of membranes of generalized Dupin cyclide type in terms of quadratures. Without loss of generality, it may be assumed that $P = P(\alpha)$ and hence $\kappa_{2\beta} = 0$. In the case $C = k^2 > 0$, the inhomogeneous linear equation (43) becomes

$$T_{zz} + k^2 T = \frac{P}{4} \tag{45}$$

with particular solution $T_0 = T_0(\alpha)$ given by

$$T_0 = \frac{1}{2k}\left(\sin(2k\alpha)\int P(\alpha)\cos(2k\alpha)\,\mathrm{d}\alpha - \cos(2k\alpha)\int P(\alpha)\sin(2k\alpha)\,\mathrm{d}\alpha\right), \tag{46}$$

while the general solution of (39) is a linear combination of

$$\phi_1 = \frac{1}{\sqrt{k}}\cos(kz), \qquad \phi_2 = \frac{1}{\sqrt{k}}\sin(kz).$$

Without loss of generality, we proceed with the one-parameter family of solutions

$$\Phi_1 = \mathrm{e}^{-\omega}\phi_1, \qquad \Phi_2 = \mathrm{e}^{\omega}\phi_2, \tag{47}$$

where ω is a real parameter. By construction, the Wronskian condition (40) is satisfied. On introduction of the scalings

$$2k\alpha \mapsto \alpha, \qquad 2k\beta \mapsto \beta, \qquad P \mapsto 2kP, \tag{48}$$

the position vector to the membranes for $\omega \neq 0$ is given by, up to a rotation,

$$\boldsymbol{r} = \frac{-c_0 F_1 + \mathfrak{b}}{a_0 \cosh\beta - c_0 \cos\alpha}\begin{pmatrix} \frac{1}{c_0}\cosh\beta \\ \sin\alpha \\ \sinh\beta \end{pmatrix} + \begin{pmatrix} -\frac{a_0}{c_0}\mathfrak{b} \\ F_2 \\ 0 \end{pmatrix}, \tag{49}$$

where

$$F_1(\alpha) = \int P(\alpha)\sin\alpha \,\mathrm{d}\alpha, \qquad F_2(\alpha) = \int P(\alpha)\cos\alpha\,\mathrm{d}\alpha \tag{50}$$

and

$$a_0 = \cosh 2\omega, \qquad c_0 = \sinh 2\omega. \tag{51}$$

It is noted that a complete generalized Dupin cyclide is obtained by specifying $F_1, F_2, c_0, \mathfrak{b}$ and smoothly matching the two surfaces corresponding to $(c_0, \mathfrak{b})$ and $-(c_0, \mathfrak{b})$ at the common boundary given by $\boldsymbol{r}(\alpha, \beta \to \pm\infty)$.

In the case $\omega = 0$, the position vector to the membranes reads

$$\boldsymbol{r} = \frac{\mathfrak{b}}{\cosh\beta}\begin{pmatrix} \cos\alpha \\ \sin\alpha \\ \sinh\beta \end{pmatrix} + \begin{pmatrix} -F_1 \\ F_2 \\ 0 \end{pmatrix}. \tag{52}$$

It will readily be seen (cf (61)) that in this case the generalized Dupin cyclide is a canal surface generated by a family of spheres of constant radius (tube). It is noted that in the limiting case $C = 0$ (obtained by carefully considering the limit $k \to 0, \omega \to \infty$ in which Φ_1, Φ_2 in (47) remain finite), the position vector of the generalized Dupin cyclides adopts the form

$$\boldsymbol{r} = \frac{2\left(-\int \alpha P(\alpha)\,\mathrm{d}\alpha + \mathfrak{b}\right)}{1 + \alpha^2 + \beta^2}\begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} -\mathfrak{b} \\ \int P(\alpha)\,\mathrm{d}\alpha \\ 0 \end{pmatrix}. \tag{53}$$

4.2. The one-parameter family of spheres

In order to provide a representation which parametrizes the entire membrane, it proves convenient to introduce a new coordinate u and a new foliation parameter μ according to

$$u = \arccos(\operatorname{sech}\beta), \qquad \mu = -\mathfrak{b}. \tag{54}$$

Then, the position vector (49) may be brought into the form

$$\boldsymbol{r}(\alpha, u) = -\psi \mathrm{e}^{\varphi}\begin{pmatrix} \frac{1}{c_0} \\ \sin\alpha\cos u \\ \sin u \end{pmatrix} + \begin{pmatrix} \frac{a_0}{c_0}\mu \\ F_2 \\ 0 \end{pmatrix}, \tag{55}$$

where

$$\psi(\alpha) = c_0 F_1 + \mu, \qquad \mathrm{e}^{-\varphi} = a_0 - c_0\cos\alpha\cos u \tag{56}$$

and

$$a_0^2 - c_0^2 = 1. \tag{57}$$

Even though $u \in [0, \pi]$, it is evident that the transition $(c_0, \mathfrak{b}) \to -(c_0, \mathfrak{b})$ is formally achieved by letting $u \in [\pi, 2\pi]$. Hence, by assuming that u is arbitrary, the complete membrane is obtained. Accordingly, we may now compare the above position vector with the standard representation [29, 30]

$$\boldsymbol{r}(s, v) = \boldsymbol{\gamma} - r r_s \boldsymbol{t} + r\sqrt{1 - r_s^2}(\cos v\boldsymbol{n} + \sin v\boldsymbol{b}) \tag{58}$$

of a canal surface, where $\boldsymbol{\gamma}(s)$ is the curve formed by the centres of the one-parameter family of spheres of radius $r(s)$ and the orthonormal triad $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ is the associated Serret–Frenet frame [10] composed of the unit tangent $\boldsymbol{t}$ to $\boldsymbol{\gamma}$, the principal normal $\boldsymbol{n}$ and the binormal $\boldsymbol{b}$. Here, s is an arc length parameter of the curve $\boldsymbol{\gamma}$ and the variable v parametrizes the circular lines of curvature $\alpha = \text{const}$. It is not difficult to show that, in the present situation, the curve $\boldsymbol{\gamma}$ is planar and given by

$$\boldsymbol{\gamma} = (-a_0 F_1(\alpha), F_2(\alpha), 0)^{\mathsf{T}} \tag{59}$$

with corresponding arc length

$$s = \int P(\alpha)\sqrt{1 + c_0^2 \sin^2\alpha}\, \mathrm{d}\alpha \tag{60}$$

by virtue of the definitions (50). The radius of the spheres is

$$r = |c_0 F_1(\alpha) + \mu| \tag{61}$$

and the curvature coordinate v is defined by the compatible pair

$$\sin v = -\frac{\sqrt{1 + c_0^2\sin^2\alpha}\sin u}{a_0 - c_0\cos\alpha\cos u}, \qquad \cos v = \frac{a_0\cos u - c_0\cos\alpha}{a_0 - c_0\cos\alpha\cos u}. \tag{62}$$

For completeness, it is noted that the Serret–Frenet frame $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ associated with $\boldsymbol{\gamma}$ consists of

$$\boldsymbol{t} = \frac{1}{\sqrt{1 + c_0^2\sin^2\alpha}}\begin{pmatrix} -a_0\sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix}, \qquad \boldsymbol{n} = -\frac{1}{\sqrt{1 + c_0^2\sin^2\alpha}}\begin{pmatrix} \cos\alpha \\ a_0\sin\alpha \\ 0 \end{pmatrix} \tag{63}$$

and $\boldsymbol{b} = (0, 0, 1)^{\mathsf{T}}$.

4.3. The generating curve

Since a canal surface is uniquely defined by its generating curve and the radius of the associated one-parameter family of spheres, the canal surfaces considered here may be recovered by studying the generating curve $\boldsymbol{\gamma}$. In fact, any planar (regular) curve

$$\boldsymbol{\gamma}(t) = (\mathsf{x}(t), \mathsf{y}(t), 0)^{\mathsf{T}} \tag{64}$$

and an associated one-parameter family of spheres with radius

$$r(t) = |\tanh(2\omega)\mathsf{x}(t) - \mu|, \qquad \omega, \mu = \text{const} \tag{65}$$

define a canal surface on which the lines of curvature are planar. Indeed, any plane curve may (locally) be obtained from (59) by a suitable choice of the parameter α and the function $P(\alpha)$. Specifically, since $F_1(\alpha)$ and $F_2(\alpha)$ are only defined up to constants of integration, it is required that

$$-c_0 P(\alpha)\sin\alpha\dot{\alpha} = \dot{\mathsf{x}}, \qquad P(\alpha)\cos\alpha\dot{\alpha} = \dot{\mathsf{y}}. \tag{66}$$

Elimination of P then shows that the relation between the parameters α and t is given by

$$\cot\alpha = -a_0\frac{\dot{\mathsf{y}}}{\dot{\mathsf{x}}} \tag{67}$$

and hence

$$P = \pm\frac{\left(\dot{\mathsf{x}}^2 + a_0^2\dot{\mathsf{y}}^2\right)^{\frac{3}{2}}}{a_0^2(\dot{\mathsf{x}}\ddot{\mathsf{y}} - \ddot{\mathsf{x}}\dot{\mathsf{y}})}, \tag{68}$$

wherein the sign has to be chosen appropriately.

Various features of the curve (59) may be retrieved from the function $P(\alpha)$. If we assume that

$$P(\alpha) = P_0 + \sum_{n=2}^{N}(\mathsf{a}_n\cos(n\alpha) + \mathsf{b}_n\sin(n\alpha)) \tag{69}$$

$(\mathsf{a}_n, \mathsf{b}_n, P_0 \in \mathbb{R})$ then γ constitutes a closed curve since $\int_0^{2\pi} P(\alpha)\sin\alpha\,\mathrm{d}\alpha = \int_0^{2\pi} P(\alpha)\cos\alpha\,\mathrm{d}\alpha = 0$ and if $P(\alpha) \neq 0$ then γ does not exhibit any cusps since $F_1'(\alpha) \sim F_2'(\alpha) \sim P(\alpha)$. It is noted that this may always be achieved by means of a suitable choice of the constant P_0. For example, the function $P(\alpha) = -\frac{3}{2}\sin 2\alpha$ generates the scaled[5] astroid $\gamma = (a_0\sin^3\alpha, \cos^3\alpha)^{\mathsf{T}}$, $\alpha \in [0, 2\pi)$ with four cusps and no self-intersections. Alternatively, the sum (69) may be replaced by an infinite series to represent a Fourier expansion of $P(\alpha)$.

4.4. *The stress resultants*

Remarkably, the stress resultants associated with the class of membranes considered here may be obtained without further integration. This is due to the existence of the first integrals

$$\mathsf{H}^{\mathsf{T}}\Lambda\mathsf{H} = I_1(\alpha), \qquad \mathsf{K}^{\mathsf{T}}\Lambda\mathsf{K} = I_2(\beta) \tag{70}$$

of the equilibrium equations $(5)_{1,2}$ so that

$$T_1 = -\frac{1}{2\kappa_2}\left(Z + \frac{I_2(\beta)}{A_2^2}\right), \qquad T_2 = -\frac{1}{2\kappa_1}\left(Z + \frac{I_1(\alpha)}{A_1^2}\right). \tag{71}$$

The remaining equilibrium equation (6) then reduces to

$$I_1 + I_2 + ZP^2 = 0 \tag{72}$$

and hence

$$I_1(\alpha) = -ZP^2(\alpha) - 2I_0, \qquad I_2(\beta) = 2I_0, \tag{73}$$

where I_0 is a constant of separation parametrizing the one-parameter family of stress distributions according to

$$T_1 = -\frac{Z}{2\kappa_2} - \frac{I_0}{\kappa_2 A_2^2}, \qquad T_2 = -\frac{Z}{2\kappa_1}\left(1 - \frac{P^2}{A_1^2}\right) + \frac{I_0}{\kappa_1 A_1^2}. \tag{74}$$

For completeness, it is observed that, in terms of the coordinates (α, u), the geometric quantities of the membrane (55) are given by

$$A_1 = P - \psi\,\mathrm{e}^{\varphi}\cos u, \qquad A_2 = -\psi\mathrm{e}^{\varphi},$$
$$\kappa_1 = \frac{\cos u}{\psi\cos u - P\,\mathrm{e}^{-\varphi}}, \qquad \kappa_2 = \frac{1}{\psi} \tag{75}$$

with ψ and φ defined by (56).

[5] In view of the factor a_0 in (59), it is more convenient to obtain curves with scaled x-coordinate: these curves are termed 'scaled'.

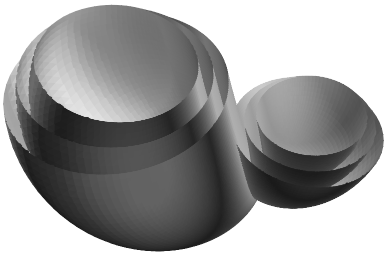

Figure 1. A family of parallel surfaces for $P(\alpha)$ defined by (79).

5. Particular membranes

In order to illustrate the preceding analysis, we now investigate the geometry of the generalized Dupin cyclides associated with canonical choices of both the function $P(\alpha)$ and the generating curve γ.

5.1. Ellipses

Here, we consider the simplest closed generating curve γ, namely the ellipse

$$\frac{\mathsf{x}^2}{w_1^2} + \frac{\mathsf{y}^2}{w_2^2} = 1, \qquad \mathsf{z} = 0 \tag{76}$$

with semi-axes w_1 and w_2. For any given parameter ω, the radius of the spheres which generate the family of parallel canal surfaces is given by (61), that is

$$r = |\tanh(2\omega)\mathsf{x} - \mu|. \tag{77}$$

If $P(\alpha) = \text{const}$ then the generating curve

$$\gamma = (a_0 P\cos\alpha, P\sin\alpha, 0)^{\mathsf{T}} \tag{78}$$

is indeed an ellipse with semi-axes $w_1 = a_0 P$ and $w_2 = P$ and eccentricity $\varepsilon = |c_0/a_0|$. However, it is evident that the parameter ω is not arbitrary but $|\tanh(2\omega)| = \varepsilon$. As indicated in section 3, this choice of P corresponds to the class of classical Dupin cyclides.

In order to obtain the complete family of canal surfaces associated with the ellipse (76), it is required to make the choice

$$P(\alpha) = \frac{a_0 w_1^2 w_2^2}{\left(w_1^2\cos^2\alpha + a_0^2 w_2^2\sin^2\alpha\right)^{\frac{3}{2}}} \tag{79}$$

and, in general, the surfaces do not constitute Dupin cyclides. It is noted that the function (79) may be represented by the series (69) with $N \to \infty$. The details are given in appendix A. A set of parallel membranes for $c_0 = 2$, $w_1 = 3$, $w_2 = 4$ is depicted in figure 1.

5.2. Talbot's curve

In the case $N = 2$, one may set

$$P(\alpha) = P_0 + 3\epsilon\cos 2\alpha, \qquad P_0 = \text{const}, \qquad \epsilon = \text{const} \tag{80}$$

without loss of generality. The generating curve (59) then becomes Talbot's curve [31]

$$\gamma = (a_0(P_0 - 2\epsilon + \epsilon\cos 2\alpha)\cos\alpha, (P_0 + 2\epsilon + \epsilon\cos 2\alpha)\sin\alpha, 0)^{\mathsf{T}}. \tag{81}$$

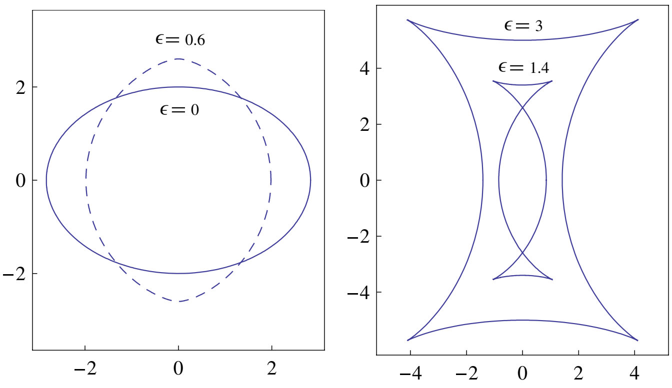

Figure 2. Talbot's curve (81) for $a_0 = \sqrt{2}$, $P_0 = 2$ and different parameters ϵ.

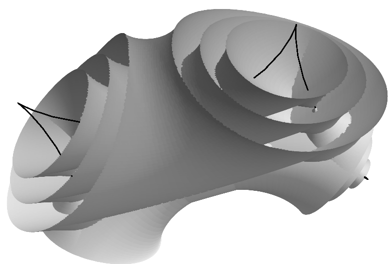

Figure 3. A family of parallel surfaces for $P(\alpha) = 2 + 9\cos 2\alpha$.

The curve (81) for $a_0 = \sqrt{2}$, $c_0 = 1$, $P_0 = 2$ and various values of the parameter ϵ are shown in figure 2. Its curvature $\varkappa$ is readily shown to be

$$\varkappa = \frac{a_0}{(P_0 + 3\epsilon\cos 2\alpha)\left(1 + c_0^2 \sin^2\alpha\right)^{\frac{3}{2}}} \tag{82}$$

and for $|P_0| > |3\epsilon|$ the curve has no cusps. If $|P_0| < |3\epsilon|$ then the curve exhibits four cusps. A family of parallel canal surfaces with Talbot's curve as the generating curve is shown in figure 3.

5.3. L-minimal generalized Dupin cyclides

An L-minimal surface [26] may be defined by the requirement that the ratio of the Gaussian curvature $\mathcal{K} = \kappa_1\kappa_2$ and mean curvature $\mathcal{H} = (\kappa_1 + \kappa_2)/2$ satisfy the Laplace equation

$$\Delta_{\rm III}\left(\frac{\mathcal{H}}{\mathcal{K}}\right) = 0 \tag{83}$$

with respect to the third fundamental form (20). Substitution of κ_1 and κ_2 from (75) into the latter equation leads to

$$P_{\alpha\alpha} + P = 0. \tag{84}$$

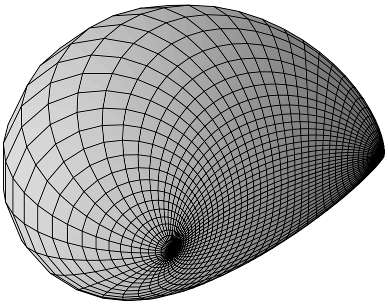

Figure 4. A closed L-minimal generalized Dupin cyclide (87) with $c_0 = \frac{3}{2}$.

Hence, $P = \cos(\alpha - \alpha_0)$ without loss of generality. Consequently, the position vector of L-minimal generalized Dupin cyclides adopts the form

$$\boldsymbol{r} = -\frac{c_0\,[2\alpha\sin\alpha_0 - \cos(2\alpha - \alpha_0)] + 4\mu}{4(a_0 - c_0\cos\alpha\cos u)}\begin{pmatrix}\frac{1}{c_0}\\ \sin\alpha\cos u\\ \sin u\end{pmatrix} + \frac{1}{4}\begin{pmatrix}\frac{4a_0}{c_0}\mu\\ 2\alpha\cos\alpha_0 + \sin(2\alpha - \alpha_0)\\ 0\end{pmatrix}. \tag{85}$$

Insertion of $F_1(\alpha)$ and $F_2(\alpha)$ as given by (50) into (59) shows that these canal surfaces are generated by the scaled cycloid

$$\gamma = \tfrac{1}{4}(-a_0(2\alpha\sin(\alpha_0) - \cos(2\alpha - \alpha_0)), 2\alpha\cos(\alpha_0) + \sin(2\alpha - \alpha_0), 0)^{\mathsf{T}}. \tag{86}$$

By construction, the surfaces (85) are both L-isothermic and L-minimal and may also be obtained from the Weierstrass-type representation proposed in [28] (cf appendix B).

It is interesting to note that there exists a sub-class of closed surfaces even though $P(\alpha)$ is not of the form (69). Thus, the one-parameter family of closed L-minimal canal surfaces

$$\boldsymbol{r} = \frac{1}{a_0 + c_0\sin\alpha\cos u}\begin{pmatrix}\sin^2\alpha\\ -a_0\sin\alpha\cos\alpha\\ c_0\sin^2\alpha\sin u\end{pmatrix} + \begin{pmatrix}0\\ \alpha\\ 0\end{pmatrix}, \qquad \alpha \in [0, \ \pi], \qquad u \in [0, 2\pi] \tag{87}$$

is obtained from (85) by appropriately shifting α and setting

$$\alpha_0 = 0, \qquad \mu = -\frac{c_0}{4}. \tag{88}$$

The Gaussian curvature of these surfaces is given by

$$\mathcal{K} = -\frac{4\cos u}{c_0\sin^3\alpha(c_0\sin\alpha\cos u + 2a_0)} \tag{89}$$

and they all exhibit cusps at the points $(0, 0, 0)$ and $(0, \pi, 0)$. A typical member of this class of membranes is depicted in figure 4. A set of such surfaces for different values of c_0 is displayed in figure 5.

6. Conclusion

We conclude this paper with two remarks. In the preceding, we have considered generalized Dupin cyclides corresponding to non-negative values of C in order to be able to focus on

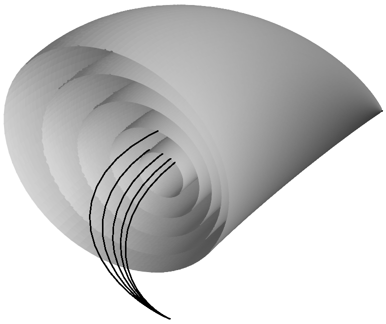

Figure 5. A family of L-minimal generalized Dupin cyclides (87) for different values of c_0.

closed generating curves and surfaces. However, the remaining case is also of interest since it includes, for instance, particular minimal surfaces. Indeed, the condition

$$\kappa_1 + \kappa_2 = -\frac{H_0}{H} - \frac{K_0}{K} = 0 \tag{90}$$

is equivalent to $R = H + K = 0$ and hence $P = c\,\mathrm{e}^{-\theta}$ by virtue of the Mainardi–Codazzi equations (23). Hence, minimal generalized Dupin cyclides are associated with the symmetry reduction $\theta = \theta(\alpha)$ of the elliptic Liouville equation $(32)_1$. Comparison of the corresponding general solution

$$\mathrm{e}^{-\theta} = \frac{\cosh(\sigma(\alpha - \alpha_0))}{\sigma} \tag{91}$$

with expression (38) indeed shows that $C < 0$ and one obtains the catenoid given by (95) with $c_0 = 0$ and α and β interchanged.

It is also interesting to note that even though not all minimal surfaces with planar lines of curvature are generalized Dupin cyclides, the latter surfaces may be mapped to all minimal surfaces with planar lines of curvature in the following sense. In section 3.1, it has been demonstrated that L-isothermic surfaces are encoded in the pair of linear equations

$$H_\beta = \theta_\beta K, \qquad K_\alpha = \theta_\alpha H, \tag{92}$$

where θ is a solution of the elliptic Liouville equation $(21)_3$. Indeed, any solution H, K of the above system uniquely defines (up to Euclidean motions) an L-isothermic surface with position vector $\boldsymbol{r}$ via integration of the pair

$$\boldsymbol{r}_\alpha = H\boldsymbol{X}, \qquad \boldsymbol{r}_\beta = K\boldsymbol{Y}, \tag{93}$$

where the unit tangent vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ are obtained from the Gauss–Weingarten equations (9) with $H_0 = K_0 = \mathrm{e}^{\theta}$ and $p = \theta_\beta, q = \theta_\alpha$. Since $\boldsymbol{X}$ and $\boldsymbol{Y}$ only depend on θ, it is evident that for any specific solution θ, the corresponding L-isothermic surfaces are, by definition, Combescure-related (see, e.g., [4]), that is they share the tangent vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$.

A particular solution of the linear system (92) is given by

$$H = \mathrm{e}^{-\theta}, \qquad K = -\mathrm{e}^{-\theta} \tag{94}$$

so that $\kappa_1 + \kappa_2 = 0$ and the corresponding surface constitutes a minimal surface. Conversely, it is easy to verify that any minimal surface is L-isothermic. Hence, L-isothermic surfaces may be regarded as Combescure transforms of minimal surfaces. Since the Combescure transformation maps planar lines of curvature to planar lines of curvature, surfaces on which the lines of curvature are planar are represented by Combescure transforms of minimal surfaces with planar lines of curvature. Accordingly, the following alternative characterization of membranes of generalized Dupin cyclide type is obtained.

Theorem 6. *Membranes on which the lines of curvature (and therefore the lines of principal stress) are planar are the membranes which may be mapped via a Combescure transformation to minimal surfaces with planar lines of curvature. All members of the latter class of minimal surfaces may be generated in this way.*

The classification of minimal surfaces which admit planar lines of curvature is classical and may be found in [10]. Thus, the class of minimal surfaces which may be mapped to the generalized Dupin cyclides (49) depends on the ratio $d_0 = c_0/a_0$ and is given by

$$\boldsymbol{r}_{\min} = \begin{pmatrix} \sqrt{1-d_0^2}\cos\alpha\cosh\beta \\ \sin\alpha\cosh\beta - d_0\alpha \\ d_0\cos\alpha\sinh\beta - \beta \end{pmatrix}. \tag{95}$$

In the case $c_0 = 0$, the catenoid is obtained. If $C = 0$ then the Enneper surface

$$\boldsymbol{r}_{\min} = \frac{1}{6}\begin{pmatrix} 3(\beta^2-\alpha^2) \\ -\alpha^3 + 3\alpha\beta^2 + 3\alpha \\ \beta^3 - 3\alpha^2\beta - 3\beta \end{pmatrix} \tag{96}$$

is the minimal surface corresponding to (53). Generalized Dupin cyclides for $C < 0$ may be mapped to the surfaces (95) with α and β interchanged.

Appendix A

For $a_0 > 0$, the function (79) can be represented by the infinite series (69), where

$$\begin{aligned} P_0 &= \frac{2}{\pi}\sqrt{\frac{w_1w_2}{a_0}}\big(2\mathbf{E}(m) - \mathbf{K}(m)\big) \\ \mathsf{a}_{2n} &= \sqrt{\frac{w_1w_2}{a_0}}\frac{(2n+1)!}{n!^2 2^{2n-1}}\left(\frac{w_2a_0 - w_1}{w_2a_0 + w_1}\right)^n {}_2\mathbf{F}_1\left(\frac{3}{2}, -\frac{1}{2}, n+1, m\right) \\ \mathsf{a}_{2n+1} &= 0, \qquad \mathsf{b}_{n+1} = 0 \\ m &= -\frac{(w_2a_0 - w_1)^2}{4w_1w_2a_0} \end{aligned}$$

for $n \geqslant 1$. Here, $\mathbf{E}(x) = \int_0^{\pi/2}(1 - x\sin^2 t)^{1/2}\mathrm{d}t$, $\mathbf{K}(x) = \int_0^{\pi/2}(1 - x\sin^2 t)^{-1/2}\mathrm{d}t$ are the complete elliptic integrals of the first and second kinds and ${}_2\mathbf{F}_1$ is a hypergeometric function.

Appendix B

A surface which is both L-isothermic and L-minimal may be represented by

$$\boldsymbol{r} = \boldsymbol{W} - \frac{\mathcal{H}}{\mathcal{K}}\boldsymbol{N}, \tag{B.1}$$

where

$$\boldsymbol{W} = \mathrm{Re}\begin{pmatrix} \int\big(-m_1 + (m_2 - \bar{m}_2)\rho + m_3\rho^2\big)F(\rho)\mathrm{d}\rho \\ \mathrm{i}\int\big(m_1 + (m_2 + \bar{m}_2)\rho + m_3\rho^2\big)F(\rho)\mathrm{d}\rho \\ \int\big(m_2 + (m_1 + m_3)\rho + \bar{m}_2\rho^2\big)F(\rho)\mathrm{d}\rho \end{pmatrix}$$

$$\frac{\mathcal{H}}{\mathcal{K}} = -\mathrm{Re}\int(m_2 - (m_1 - m_3)\rho - \bar{m}_2\rho^2)F(\rho)\mathrm{d}\rho + \mu \tag{B.2}$$

and

$$\boldsymbol{N} = -\frac{1}{1+\rho\bar{\rho}}\begin{pmatrix} \rho + \bar{\rho} \\ \mathrm{i}(\rho - \bar{\rho}) \\ 1 - \rho\bar{\rho} \end{pmatrix} \tag{B.3}$$

is a normal vector to the surface (B.1). The parameters m_1, m_3 are real, m_2 is complex and $F(\rho)$ is an arbitrary holomorphic function of the complex coordinate ρ. The substitution

$$m_1 = (a_0 + c_0)\cos\alpha_0, \qquad m_2 = \sin\alpha_0, \qquad m_3 = (c_0 - a_0)\cos\alpha_0,$$

$$F(\rho) = \frac{2}{(a_0 + c_0 + (a_0 - c_0)\rho^2)^2} \tag{B.4}$$

in the latter representation produces the set of L-minimal generalized Dupin cyclides (85) up to a rotation. The function ρ is defined in terms of curvature coordinates as

$$\rho = (a_0 + c_0)\tan\left(\frac{\alpha + \mathrm{i}\beta}{2}\right), \tag{B.5}$$

where $\beta = \operatorname{arccosh}(\sec u)$. A derivation of the Weierstrass representation of L-isothermic surfaces which are L-minimal is given in [28].

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